

# The optimization model of the heat conduction structure

Yongcun Zhang, Shutian Liu\*

*State Key Laboratory of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics,  
Dalian University of Technology, Dalian 116024, China*

Received 10 November 2007; received in revised form 14 January 2008; accepted 15 January 2008

## Abstract

An optimization model considering a novel thermal performance index to be the objective function is proposed for minimizing the highest temperature in this paper. Firstly, the performance of the conventional heat conduction optimization model, with the dissipation of heat transport potential capacity as the objective function, is evaluated by a one-dimensional heat conduction problem in a planar plate exchanger. Then, a new thermal performance index, named the geometric average temperature, is introduced. The new heat conduction optimization model, with the geometric average temperature as the objective function, is developed and the corresponding finite element formula is presented. The results show that the geometric average temperature is an ideal thermal performance index and the solution of the new model is close to the theoretical optimal solution.

© 2008 National Natural Science Foundation of China and Chinese Academy of Sciences. Published by Elsevier Limited and Science in China Press. All rights reserved.

*Keywords:* Optimization; Heat conduction structure; Dissipation of heat transport potential capacity; Geometric average temperature; Planar plate exchanger

## 1. Introduction

The progress toward smaller scales in electronics makes the cooling of the integrate circuits become an important issue. The conventional convective cooling method which is feasible and is often used to control the temperature of a system becomes impractical because the channels of heat transfer take up too much space for high compacted integrate circuit. Hence, it is necessary to build heat conduction structures with high-conductivity materials so that the heat can be collected, transferred and exchanged with external environment automatically and rapidly [1,2]. A key problem is how to design the structures with a rational distribution of high-conductivity materials, which not only benefits the temperature control but also can reduce material and manufacturing costs and bring possibilities for further miniaturization.

Designs of the optimal heat conduction structure have attracted much attention and many achievements have been made [1–24], including mathematical models and the corresponding solving methods. For example, Bejan et al. put forward a tree-like network construction method based on the constructal theory [1–8]. Guo et al. proposed some practical design criteria and developed the corresponding optimization methods for the heat conduction structure based on the least dissipation principle of heat transport potential capacity [9–15]. The topology optimization method has also been applied for the heat conduction structural optimization [16–22]. In all these cases, the nature of optimization design for the heat conduction structures is to build a mathematical model that maximizes or minimizes an objective function (e.g. the thermal performance index) subjected to certain constraints. Thus, it is a key to define a suitable thermal performance index in such an optimization model.

Statistical data show that the failure of the real devices with a fraction of 55% is caused by the high temperature,

\* Corresponding author. Tel.: +86 411 4707 057; fax: +86 411 470 8769.  
E-mail address: [stliu@dlut.edu.cn](mailto:stliu@dlut.edu.cn) (S. Liu).

and this fraction increases exponentially with the increasing temperature [25,26]. Thus, the highest temperature is a primary factor that induces the failure of practical cooling structure and should be well controlled. In practice, it is natural to define the highest temperature as an objective function of the optimization model. However, the location of the highest temperature usually changes with the change of material distribution in the optimization process and is a discontinuous function of the design variables, which may introduce numerical difficulties in optimization. Therefore, instead of a directing optimization of the highest temperature, it is more convenient to define another proper thermal performance index as the objective function in an optimization model to indirectly accomplish the goal of minimizing the highest temperature.

In the optimization model of a heat conduction structure, the objective function can be selected as

$$f(\mathbf{X}) = \int_{\Omega} \frac{1}{2} (-\mathbf{q}(\mathbf{X}) \nabla T(\mathbf{X})) d\Omega \quad (1)$$

where  $\mathbf{X}$  is the design variable used to describe the distribution of material,  $\mathbf{q}(\mathbf{X})$  is the flux density and  $\nabla T(\mathbf{X})$  is the temperature gradient. Using the finite element formulation, Eq. (1) can also be written as

$$f(\mathbf{X}) = \mathbf{T}^T \mathbf{K}(\mathbf{X}) \mathbf{T} \quad (2)$$

where  $\mathbf{T}$  is the global temperature vector and  $\mathbf{K}(\mathbf{X})$  is the thermal conductivity matrix. Generally, Eq. (1) is defined as the dissipation of heat transport potential capacity (DHTPC) [11], and the least dissipation principle of heat transport potential capacity is presented based on this definition; Eq. (2) is defined as the heat dissipation efficiency [17,18], which is the objective function of the heat conduction topology optimization.

Using the DHTPC (or the heat dissipation efficiency) as a thermal performance index, some good design results have been obtained. However, this index can only tell us the heat dissipative capability rather than the highest temperature. How big the difference between the optimal design by the optimization model with DHTPC and the present design that is for minimizing the highest temperature? Is there any better thermal performance index? Answers to these questions are the motivation of this study.

In this paper, the difference between the DHTPC and the present design goal is evaluated by a one-dimensional heat conduction problem for a planar plate exchanger. The geometric average temperature (GAT) is proposed as a new thermal performance index and the corresponding heat conduction optimization model is developed.

## 2. Heat conduction optimization of the planar plate exchanger

In many practical cooling structures, a commonly used design criterion is that the highest temperature must not exceed a specified value. However, the optimization objective in many existing heat conduction optimization models

is the DHTPC. To evaluate the quality of these exciting models, we compared their results with those obtained from an optimization model with the highest temperature as the objective function. For simplicity, the presented example is a one-dimensional heat conduction problem for a planar plate, which can be solved analytically.

### 2.1. Problem description

A rectangular planar plate exchanger, with length  $l$ , width  $W$  ( $W \gg l$ ) and thickness  $t$ , is embedded in the heater. The heat generated by a heater flows into the exchanger uniformly. The heat flowing into the exchanger is  $q''$  per unit time and area. Only one side along the width direction of the exchanger contacts with a thermostat with a constant temperature  $T_0$  and others are adiabatic. This problem can be described as a planar heat conduction model with a uniform heat source, as shown in Fig. 1. Furthermore, this model can be simplified into a one-dimensional heat transfer problem because the thickness  $t$  and the internal heat source  $q''$  do not change along the width direction. The goal is to obtain the optimal heat conduction performance by designing the thickness  $t$  along the length direction of the exchanger.

Since the thermal conductivity is proportional to the thickness  $t$ , the thickness design can be transformed into the conductivity field design. That is to say, the limitation of material,  $\int_0^l t(x) W dx = \text{const}$ , can be written as the capability of conductivity,  $\int_0^l k(x) dx = K_0$ , where  $K_0$  is a constant. The governing equation of the heat conduction in the exchanger can be described as

$$\begin{aligned} q(x) &= -k(x) \frac{dT}{dx}, \quad \frac{dq}{dx} + q'' = 0, \quad 0 < x < l \\ T(x=0) &= T_0, \quad q(x=l) = 0 \end{aligned} \quad (3)$$

where  $k(x)$  is the thermal conductivity,  $q(x)$  is the heat flux density and  $T(x)$  is the temperature. In addition, the heat flux is assumed to be positive along the  $x$  direction. Solving (3), we can obtain

$$\begin{aligned} q &= -q''(l-x), \\ T(x) &= T_0 + \int_0^x \nabla T dx = T_0 + q'' \int_0^x (l-x)/k(x) dx \end{aligned} \quad (4)$$

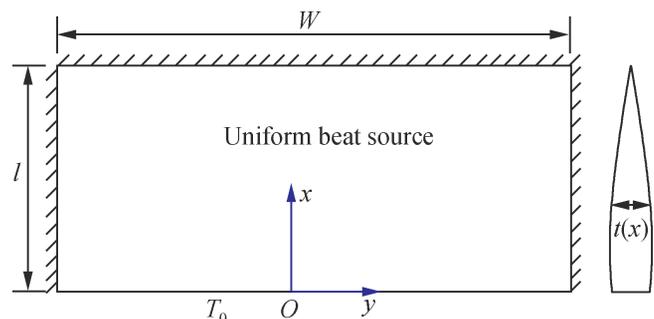


Fig. 1. A theoretical model of a planar plate exchanger.

Then, the optimization design for the exchanger is to determine the optimal heat conduction performance by designing the conductivity field under a given integral of the thermal conductivity (or material volume) over the design domain. Let  $f(k)$  denote a thermal performance index. The heat conduction optimization problem can be formulated as

$$\begin{aligned} &\text{Find : } k(x) \\ &\text{min : } f(k) \\ &\text{s.t. : } \int_0^l k(x)dx = K_0 \end{aligned} \tag{5}$$

Using the Lagrange multiplier method, the solution of the thermal conductivity field can be determined by

$$\delta_k(f(k)) + \lambda \int_0^l \delta k dx = 0, \quad \delta \lambda \left( \int_0^l k dx - K_0 \right) = 0 \tag{6}$$

### 2.2. Minimization of the highest temperature

According to the heat conduction theory, the highest temperature is located on the boundary of  $x = l$  and can be written as

$$T_{\max}(k) = T_0 + q'' \int_0^l (l-x)/k(x)dx \tag{7}$$

Substituting Eq. (7) into Eq. (6), we have

$$\int_0^l [-q''(l-x)/k^2(x) + \lambda] \delta k(x)dx = 0, \quad \int_0^l k dx = K_0 \tag{8}$$

The optimal thermal conductivity field  $k_{T_{\max}}(x)$  can be obtained by solving Eq. (8), which is

$$k_{T_{\max}}(x) = \frac{3K_0}{2l^{3/2}}(l-x)^{1/2} \tag{9}$$

and the corresponding temperature distribution is

$$T_{T_{\max}}(x) = T_0 + \frac{4q''l^{3/2}}{9K_0}(l^{3/2} - (l-x)^{3/2}) \tag{10}$$

Introducing a dimensionless parameter

$$\tilde{x} = x/l \tag{11}$$

the conductivity field and the temperature distribution can be expressed in the dimensionless space as

$$\tilde{k}_{T_{\max}}(\tilde{x}) = \frac{k(x)}{K_0/l} = 3(1-\tilde{x})^{1/2}/2 \tag{12}$$

and

$$\tilde{T}_{T_{\max}}(\tilde{x}) = \frac{T(x) - T_0}{q''l^3/K_0} = 4(1 - (1-\tilde{x})^{3/2})/9 \tag{13}$$

where subscript  $T_{\max}$  denotes that the optimization objective is to minimize the highest temperature.

### 2.3. Minimization of the dissipation of heat transport potential capacity

For the planar plate exchanger, the DHTPC can be expressed as

$$f(k) = \frac{(q'')^2}{2} \int_0^l \frac{(l-x)^2}{k} dx \tag{14}$$

where the DHTPC is considered to be an optimization objective function, the optimal thermal conductivity field should obey the following necessary conditions:

$$\int_0^l \left[ -\frac{(q'')^2(l-x)^2}{2k^2(x)} + \lambda \right] \delta k(x)dx = 0, \quad \int_0^l k dx = K_0 \tag{15}$$

The thermal conductivity field can be obtained by solving Eq. (15), which is

$$k_{\text{dis}}(x) = \frac{2K_0}{l^2}(l-x) \tag{16}$$

and the corresponding temperature distribution is

$$T_{\text{dis}}(x) = T_0 + \frac{q''l^2}{2K_0}x \tag{17}$$

The dimensionless thermal conductivity field and the temperature distribution are

$$\tilde{k}_{\text{dis}}(\tilde{x}) = \frac{k(x)}{K_0/l} = 2(1-\tilde{x}) \tag{18}$$

and

$$\tilde{T}_{\text{dis}}(\tilde{x}) = \frac{T(x) - T_0}{q''l^3/K_0} = \tilde{x}/2 \tag{19}$$

where subscript dis denotes that the optimization objective is to minimize the DHTPC.

### 2.4. Comparisons of two different optimization models

The dimensionless thermal conductivity fields and the corresponding dimensionless temperature distributions from the two different optimization models are shown in Fig. 2. To facilitate comparisons, the temperature distribution with uniformly distributed thermal conductivity (denoted by ‘av’) is analyzed, which can be expressed as

$$T_{\text{av}} = T_0 + \int_0^x \nabla T dx = T_0 + \frac{q''l(lx - x^2/2)}{K_0} \tag{20}$$

and the corresponding dimensionless temperature distribution is

$$\tilde{T}_{\text{av}} = \tilde{x} - \tilde{x}^2/2 \tag{21}$$

which is also plotted in Fig. 2. It can be found that the temperature distribution from the model with an objective function of the DHTPC has an obvious reduction in the internal exchanger when compared with the temperature field from the model with a uniform thermal conductivity

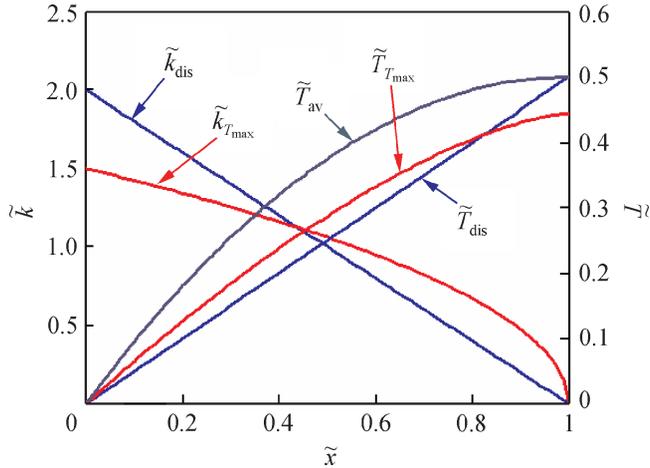


Fig. 2. Comparisons of thermal conductivity fields and temperature distributions from different optimization models. av: the uniform conductivity field; dis: the dissipation of heat transport potential capacity;  $T_{\max}$ : the highest temperature.

field. However, these two models give the same highest temperature. In addition, when compared with the model with an objective function of the highest temperature, large differences in the thermal conductivity field can be found and the highest temperature increases by 12.5%, which indicates that the optimization model with an objective function of the DHTPC sometimes cannot fulfill the present design goal. Thus, it is necessary to propose new thermal performance indexes for the optimization model.

### 3. Optimization model based on the geometric average temperature

#### 3.1. Objective function and optimization model

As mentioned above, the optimal design by the optimization model with the DHTPC as an objective function sometimes introduces large errors compared with the present design goal. Furthermore since the highest temperature is a discontinuous function of the design variables, direct optimization of it will bring numerical difficulties. To achieve a good tradeoff between the optimization performance and numerical cost, a new thermal performance index called the geometric average temperature  $T_{\text{geoav}}$  is proposed, which can be expressed as

$$T_{\text{geoav}} = \left( \frac{1}{|\Omega|} \int_{\Omega} (T(x))^n dx \right)^{1/n}, x \in \Omega \quad (22)$$

where  $|\Omega|$  denotes the area (or volume) over the design region. Theoretically, the geometric average temperature is close to the highest temperature when  $n$  is infinitely large, i.e.  $T_{\text{geoav}} \xrightarrow{n \rightarrow \infty} T_{\max}$ . Thus, the geometric average temperature is an appropriate approximation of the highest temperature. The new heat conduction optimization model can be written as

$$\text{Find : } X = k(x), \quad x \in \Omega$$

$$\text{min : } T_{\text{geoav}}(X) = \left( \frac{1}{|\Omega|} \int_{\Omega} (T(x))^n dx \right)^{1/n} \quad (23)$$

$$\text{s.t. : } \int_{\Omega} k d\Omega = K_0, \quad K_0 = \text{const}$$

Here, the finite element method is used to solve the optimization problem. Suppose that the material is uniformly distributed and has the same conductivity in each element, then the distribution of the material can be described by different thermal conductivities in each element mesh, which can be expressed by the finite element method

$$k(x) = k_e, \quad x \in \Omega_e, \quad e = 1, 2, \dots, Ne \quad (24)$$

where  $k_e$  ( $e = 1, 2, \dots, Ne$ ) denotes the thermal conductivity of the  $e$ -th element,  $\Omega_e \in \Omega$  the region of the  $e$ -th element and  $Ne$  the total number of elements. Then, the temperature and its  $n$  power in an element can be written as

$$T(x) = [N(x)]\{\bar{T}\} \quad T^n(x) = [N(x)]\{\bar{T}^n\} \quad (25)$$

$$\{\bar{T}\} = (T_1, T_2, \dots, T_{N_p})^T,$$

$$\{\bar{T}^n\} = [(T_1)^n, (T_2)^n, \dots, (T_{N_p})^n]^T \quad (26)$$

where  $\{\bar{T}\}$  and  $\{\bar{T}^n\}$  denote the temperature vector of nodes and the corresponding  $n$  power, respectively.  $T_n$  ( $n = 1, 2, \dots, N_p$ ) is the temperature of the  $n$ -th node,  $N_p$  is the total number of nodes and  $[N(x)]$  is the shape function matrix. The node temperature can be solved by the following governing equation:

$$[K]\{\bar{T}\} = \{Q\} \quad (27)$$

Here,  $\{Q\}$  is the thermal flux vector and  $[K]$  is the thermal conductivity matrix which can be assembled by the element thermal conductivity matrix

$$[K] = \sum_{e=1}^{Ne} [K_e], \quad [K_e] = k_e [K_e^0] \quad (28)$$

where  $[K_e^0]$  is the  $e$ -th element thermal conductivity matrix with a unit thermal conductivity.

The geometric average temperature can be rewritten as

$$T_{\text{geoav}} = ([B]\{\bar{T}^n\})^{1/n}, \quad [B] = \frac{1}{|\Omega|} \int_{\Omega} [N(x)] dx \quad (29)$$

Therefore, the heat conduction optimization problem can be expressed as

$$\text{Find : } X = (k_1, k_2, \dots, k_{Ne})^T$$

$$\text{min : } T_{\text{geoav}}(X) = ([B]\{\bar{T}^n\})^{1/n} \quad (30)$$

$$\text{s.t. : } \sum_{e=1}^{Ne} k_e V_e = K_0, \quad V_e = |\Omega_e|$$

The feasible direction method was employed to find the optimal solution. The sensitivity of the objective function (the geometric average temperature) can be expressed as

$$\frac{\partial T_{\text{geoav}}}{\partial k_e} = \frac{1}{n} ([B]\{\bar{T}^n\})^{-(n-1)/n} [B] \frac{\partial}{\partial k_e} \{\bar{T}^n\} \quad (31)$$

where

$$\frac{\partial}{\partial k_e} \{\bar{T}^n\} = \text{diag}(nT_1^{(n-1)}, nT_2^{(n-1)}, \dots, nT_{N_p}^{(n-1)}) \left\{ \frac{\partial \bar{T}}{\partial k_e} \right\} \quad (32)$$

and

$$\left\{ \frac{\partial \bar{T}}{\partial k_e} \right\} = -[K]^{-1} \left( \frac{\partial}{\partial k_e} [K] \right) \{\bar{T}\} = -[K]^{-1} [K_e^0] \{\bar{T}\} \quad (33)$$

### 3.2. Results and discussion

The planar plate exchanger is analyzed again by the new optimization model (23), in which the one-dimensional heat conduction element with two the nodes was used to mesh the design domain and the feasible direction method was employed to find the optimal solution. The obtained thermal conductivity field (material distribution) and the corresponding temperature distribution are shown in Fig. 3. To facilitate comparisons, the solutions of the optimization models with the DHPC and the highest temperature as objective functions are also shown in Fig. 3. It can be seen that the thermal conductivity field and the temperature distribution are close to the present design goal (the solution of minimizing the highest temperature) when the power index  $n$  is larger than 16. Thus, the geometric average temperature is an ideal thermal performance index. The new optimization model with the geometric average temperature as the objective function is a more accurate description for the design goal than that with the DHTPC as the objective function. The results obtained from the convective optimization model with the DHTPC as the objective function is equal to that from the new optimization model when the power index  $n$  is 1. With the increasing of power index  $n$ , the thermal conductivity field and the temperature field obtained by the new model is rapidly

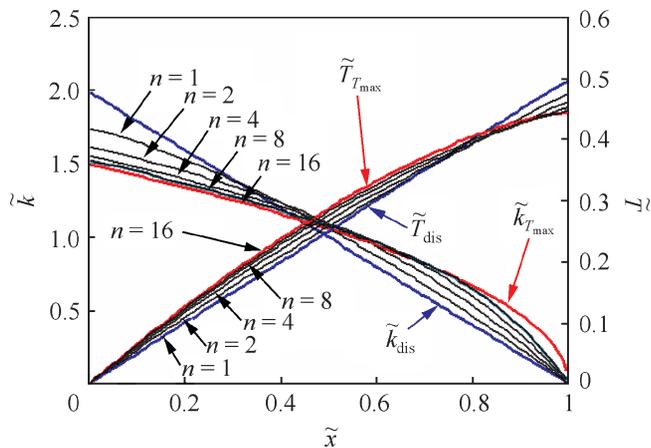


Fig. 3. Solutions of the optimization model based on the geometric average temperature with different power indexes. Results from other models are also shown for comparison.

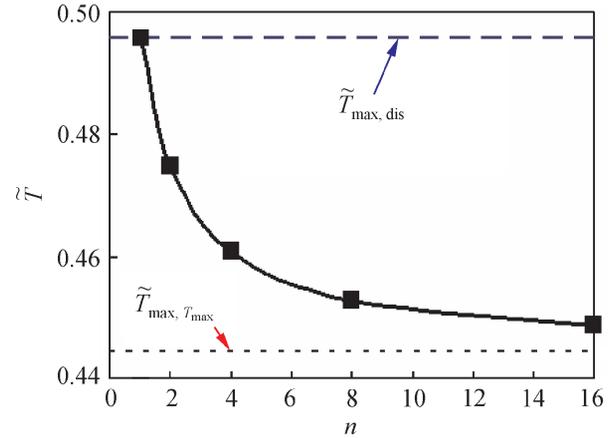


Fig. 4. The highest temperature obtained by the optimization model based on the geometric average temperature versus the power index  $n$ .  $\tilde{T}_{\text{max,dis}}$  and  $\tilde{T}_{\text{max},T_{\text{max}}}$  denote the corresponding highest temperatures of the optimal solution of optimization models with the DHTPC and the highest temperature as the objective function, respectively.

close to the ideal design. The change of the corresponding highest temperature with the increasing power index  $n$  is shown in Fig. 4. Since the approximate level tends to stabilize with the increasing power index  $n$ , an appropriate value is required to select for the power index  $n$  in a practical optimization process.

### 4. Conclusions

In this paper, we have discussed how to minimize the highest temperature of a heat conduction structure by designing the material distribution with a specified material volume (conductivity ability). The large error sometimes occurs between the results by the usual optimization model with an objective function of the DHTPC and the theoretical optimal design. A geometric average temperature has been proposed, which is a better thermal performance index as the objective function. The solution of the new model with the geometric average temperature as the objective function is close to the theoretical optimal solution.

### Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant Nos. 10332010, 10721062), and the National Key Basic Research Program of China (Grant No.2006CB601205) and the Program for New Century Excellent Talents in the University of China (2004).

### References

- [1] Bejan A. Constructal theory network of conducting paths for cooling paths for cooling a heat generating volume. *Int J Heat Mass Tran* 1997;40(4):799–816.
- [2] Bejan A. Shape and structure, from engineering to nature. Cambridge: Cambridge University Press; 2000.

- [3] Ledezma GA, Bejan A, Errera MR. Constructal tree network for heat transfer. *J Appl Phys* 1997;82(1):89–100.
- [4] Almgogbel M, Bejan A. Conduction trees with spacing at tips. *Int J Heat Mass Tran* 1999;42(20):3739–56.
- [5] Almgogbel M, Bejan A. Constructal optimization of nonuniformly distributed tree-shaped flow structures for conduction. *Int J Heat Mass Tran* 2001;44(22):4185–94.
- [6] Rocha LAO, Lorente S, Bejan A. Constructal design for cooling a disc-shaped area by conduction. *Int J Heat Mass Tran* 2002;45(8):1643–52.
- [7] Silva AKD, Vasile C, Bejan A. Disc cooling with high-conductivity inserts that extend inward from the perimeter. *Int J Heat Mass Tran* 2004;47(19–20):4257–63.
- [8] Rocha LAO, Lorente S, Bejan A. Conduction tree networks with loops for cooling a heat generating volume. *Int J Heat Mass Tran* 2006;49(15–16):2626–35.
- [9] Cheng XG, Xia ZZ, Li ZX, et al. Optimization of heat conduction: thermal dissipation and optimal thermal conductivity distribution. *J Eng Thermophys* 2002;23(6):715–7, [in Chinese].
- [10] Xia ZZ, Guo ZY. Heat conduction optimization simulated by life evolution. *Progr Nat Sci* 2001;11(8):845–52, [in Chinese].
- [11] Guo ZY, Cheng XG, Xia ZZ. Least dissipation principle of heat transport potential capacity and its application in heat conduction optimization. *Chin Sci Bull* 2003;48(4):406–10.
- [12] Cheng XG, Li ZX, Guo ZY. Variational principles in heat conduction. *J Eng Thermophys* 2004;25(3):457–9, [in Chinese].
- [13] Guo ZY, Liang XG, Zhu HY. Entranspy – A new physical property of the overall capacity of heat transport. *Progr Nat Sci* 2006;16(10):1288–96, [in Chinese].
- [14] Cheng XG, Meng JA, Guo ZY. Potential capacity dissipation minimization and entropy generation minimization in heat conduction optimization. *J Eng Thermophys* 2005;6(6):1034–6, [in Chinese].
- [15] Han GZ, Guo ZY. Two different thermal optimization objective functions: dissipation of heat transport potential capacity and entropy production. *J Eng Thermophys* 2006;27(5):811–3, [in Chinese].
- [16] Li Q, Steven GP, Xie YM, et al. Evolutionary topology optimization for temperature reduction of heat conducting fields. *Int J Heat Mass Tran* 2004;47(23):5071–83.
- [17] Jia HP. Topology optimization of structure and compliant mechanism. Ph.D Dissertation, Dalian University of Technology, 2004. p. 95–118, [in Chinese].
- [18] Zuo KT, Chen LP, Zhang YQ, et al. Structural optimal design of heat conductive body with topology optimization method. *Chin J Mech Eng* 2005;41(4):13–21, [in Chinese].
- [19] Zhang YC, Liu ST. Design of conducting paths based on topology optimization. *Heat Mass Tran*, online; doi:10.1007/s00231-007-0365.
- [20] Li JC, Ye BY, Tang Y, et al. Algorithm of topology optimization criteria for heat conduction structure based on density approach. *J South China Univ Technol (Nat Sci Ed)* 2006;34(2):27–32, [in Chinese].
- [21] Xu XH, Liang XG, Xun RJ. Optimization of heat conduction using combinatorial optimization algorithms. *Int J Heat Mass Tran* 2007;50(9–10):1675–82.
- [22] Francois MP, Louis G. Optimal conduction pathways for a heat generating body: a comparison exercise. *Int J Heat Mass Tran* 2007;50(15–16):2996–3006.
- [23] Wu WJ, Chen LG, Sun FR. Improvement of tree-like network constructal method for heat conduction optimization. *Sci China E Technol Sci* 2006;49(3):1–10.
- [24] Zhou SB, Chen LG, Sun FR. Optimization of constructal volume-point conduction with variable cross section conducting path. *Energy Conver Manag* 2007;48(1):106–11.
- [25] Liu MA, Dong QX, Chen S. Investigation on thermal design for opening cabinet of electronic equipment by numerical simulation. *J Air Force Eng Univ (Nat Sci Ed)* 2005;6(2):62–5, [in Chinese].
- [26] Qi YQ, He YL, Zhang W, et al. Thermal analysis and design of electronic equipments. *Mod Electron Technol* 2003;144:73–6, [in Chinese].